# The Orbifold Notation for Surface Groups

inspired by John Conway and Bill Thurston

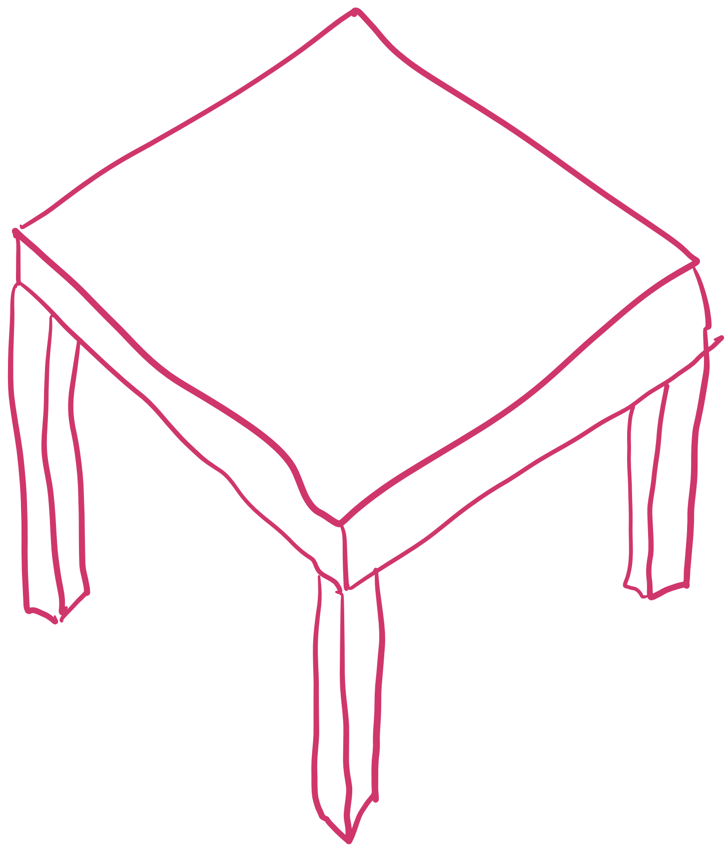
Savannah Crawford - UofU Pre-REU

## A philisophical question

is there a limit to the number of ways the Euclidean plane ($\mathbb{R}^2$) can be tiled?

What if I equate patterns with the same symmetries?

# Symmetries of a Table



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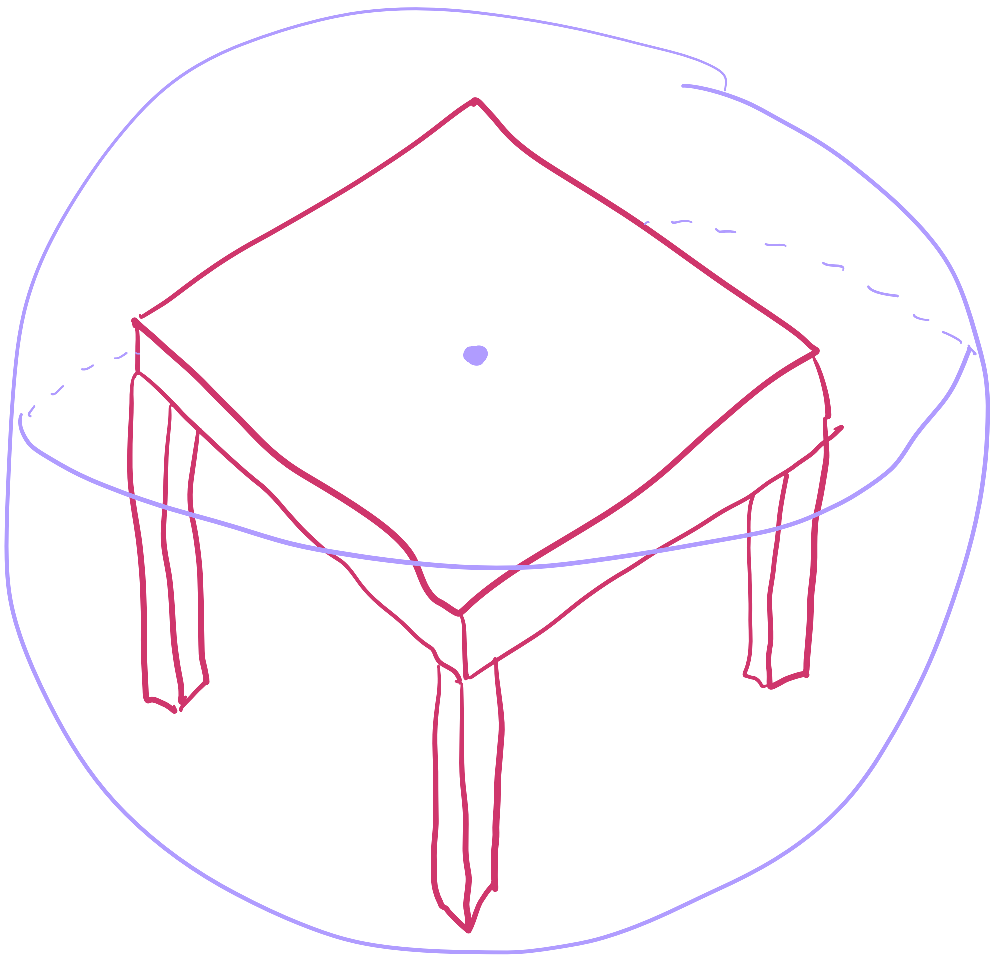
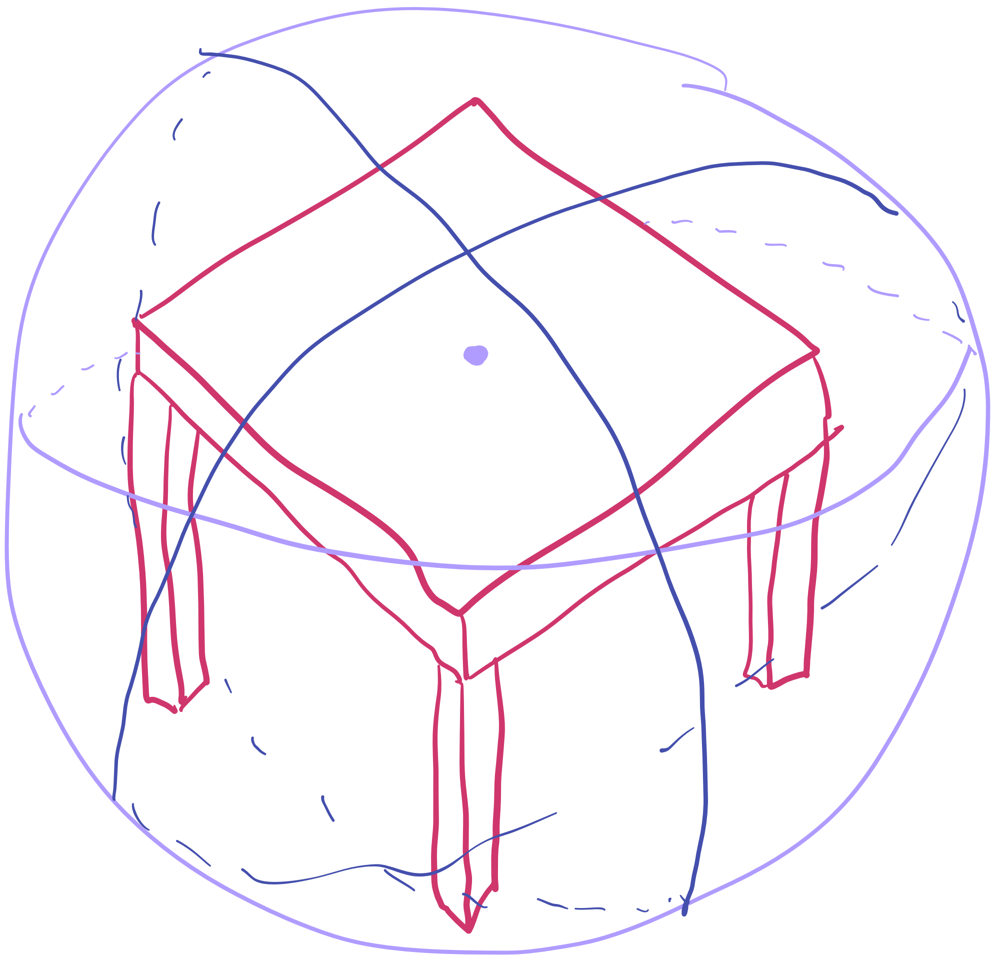


# Symmetries of a Table



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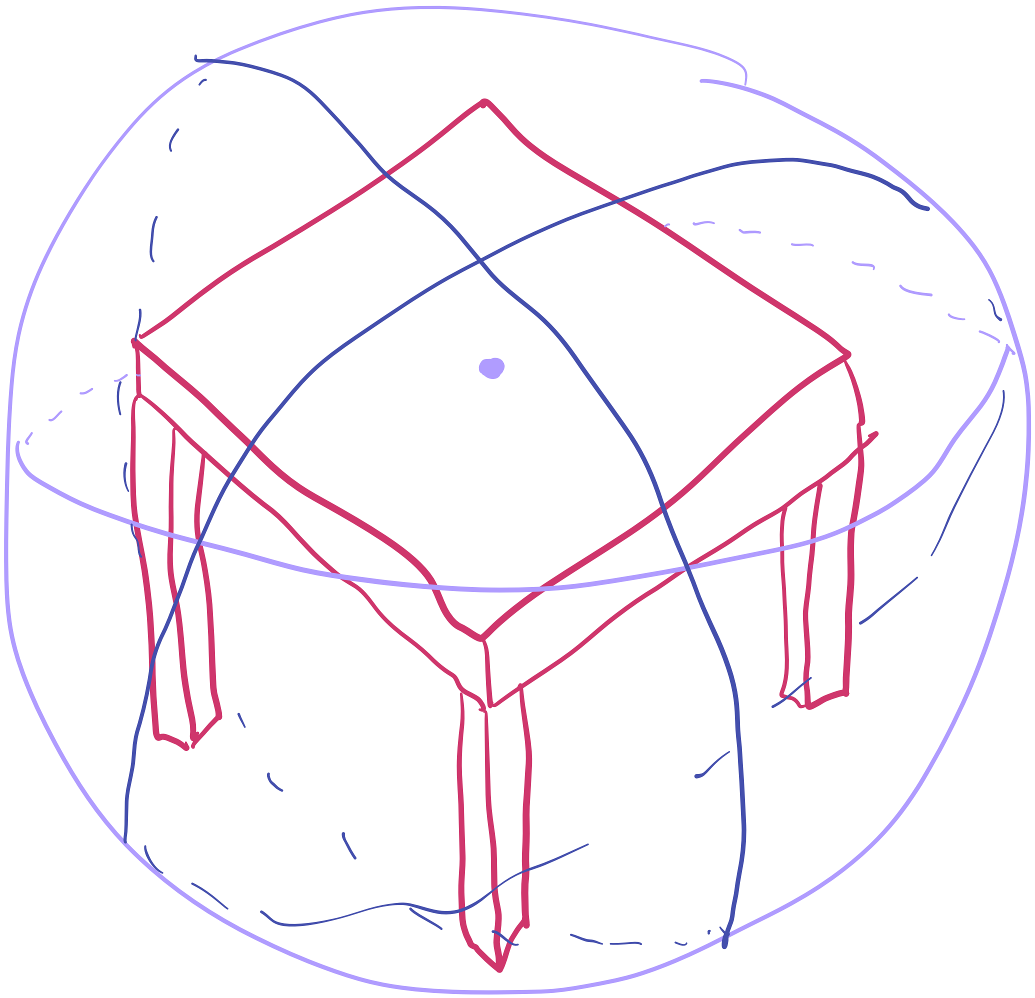
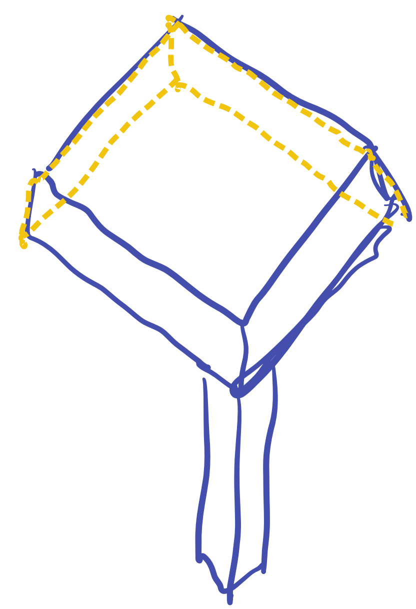


The orbifold of a table



of a table

# Definitions

**topology** a field of mathematics which is concerned with the properties of a geometric object which are preserved under continuous defermations, such as stretching, twisting, crumpling and bending.

# Definitions

**isometry** a bijective map between two metric spaces which preserves distances

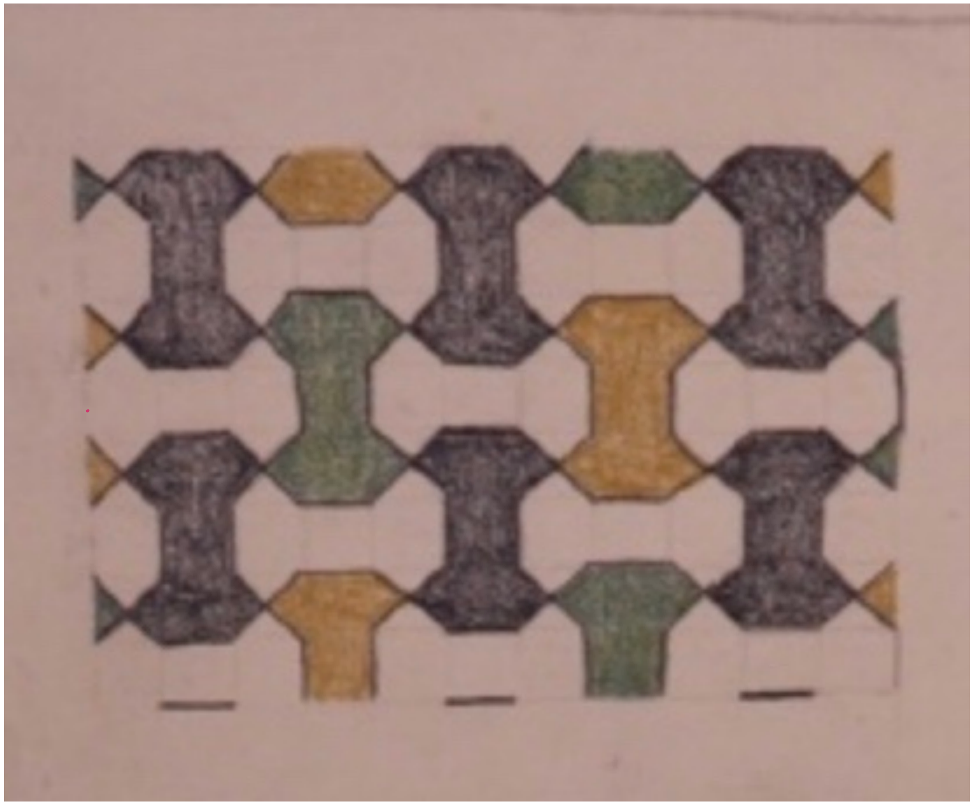
$f: \mathbb{R}^2 \to \mathbb{R}^2$

ie $\forall\, x, y \in \mathbb{R}^2$

then $d(x,y) = d(f(x), f(y))$

**wallpaper group** a discrete group of isometries of the Euclidean plane

# Classifying a Wallpaper pattern



1. Identify mirror lines

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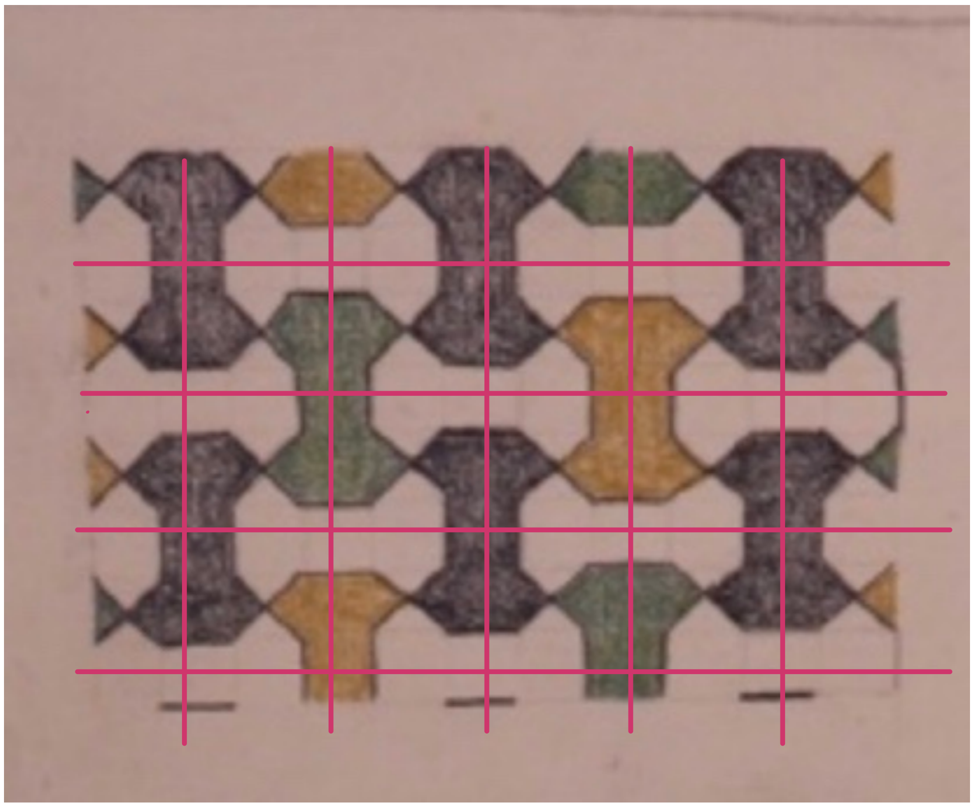

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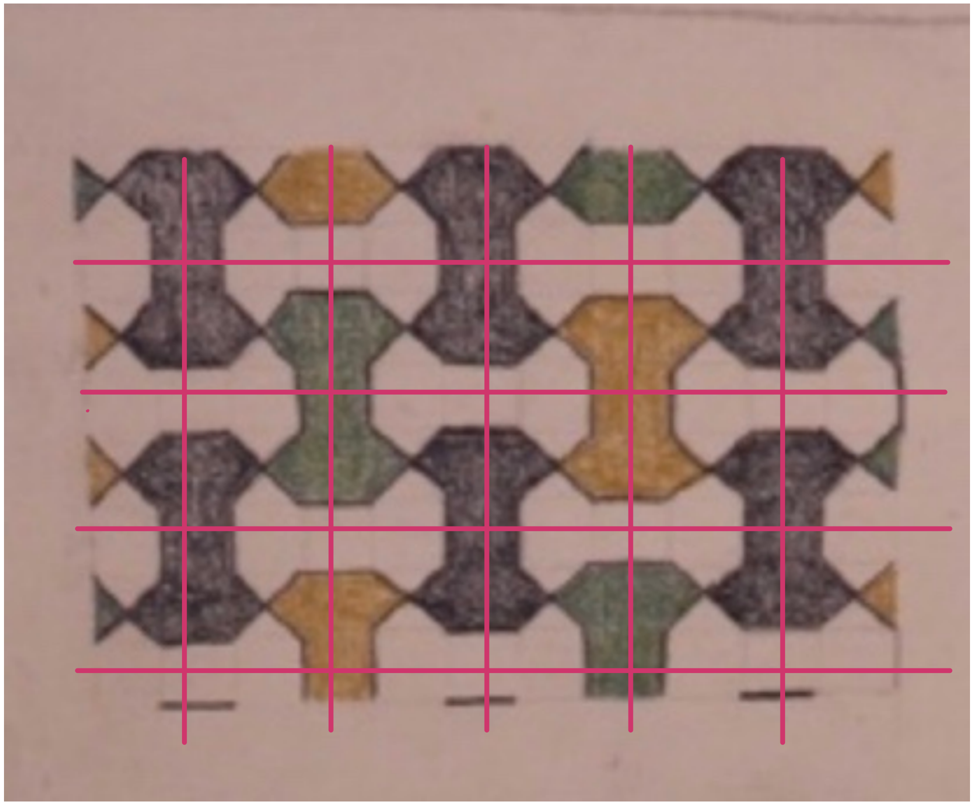

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2. identify points of rotation not on mirror lines

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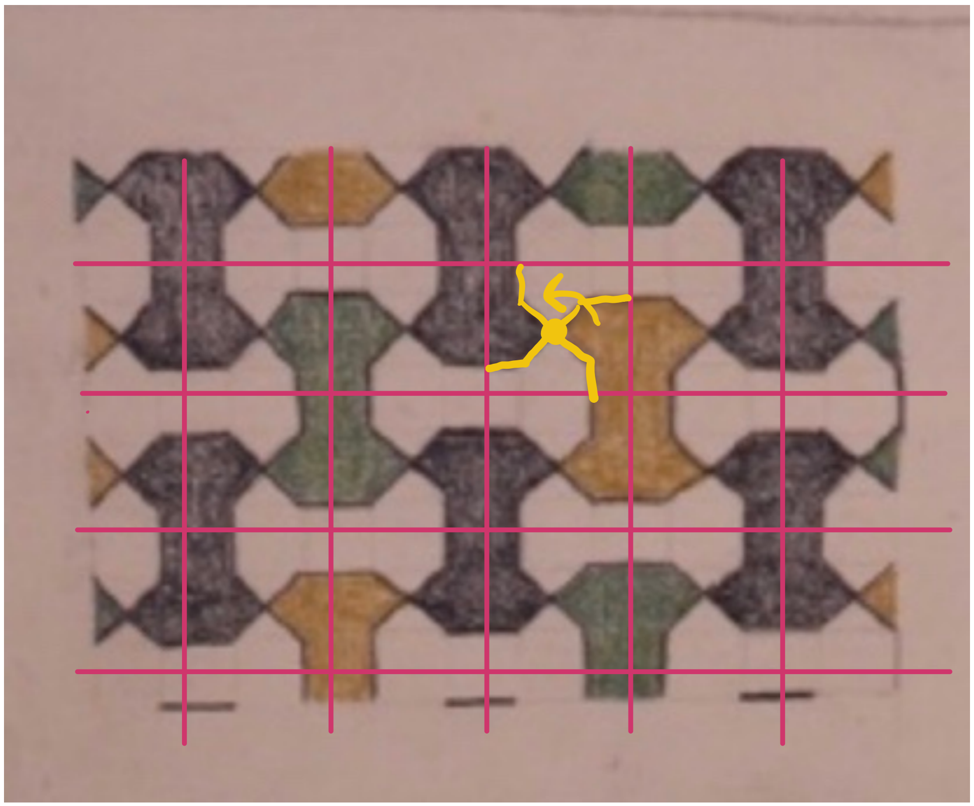

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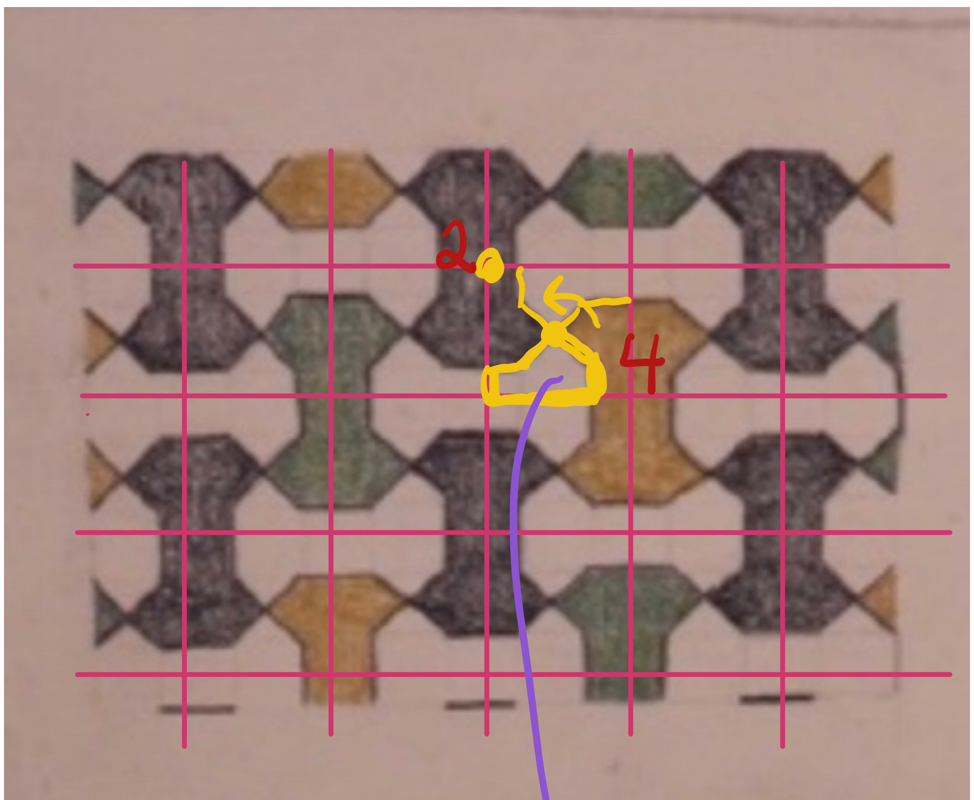


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2. identify points of rotation <u>not</u> on mirror lines

$\boxed{4*2}$

"fundamental domain"

# Symmetries in $\mathbb{R}^2$

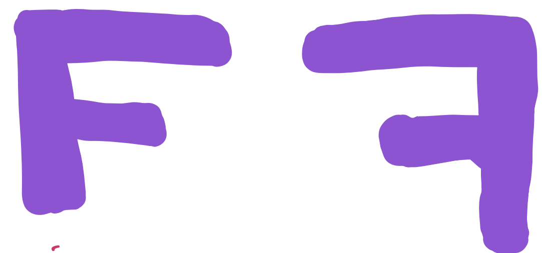
→ isometries respect Euclidean distance

ie $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

Translation

wonder

Reflection

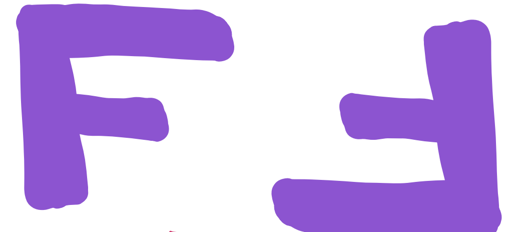


mirror

Glide Reflection

kaleidoscopes

Rotation



gyration

# Understanding Mirrors

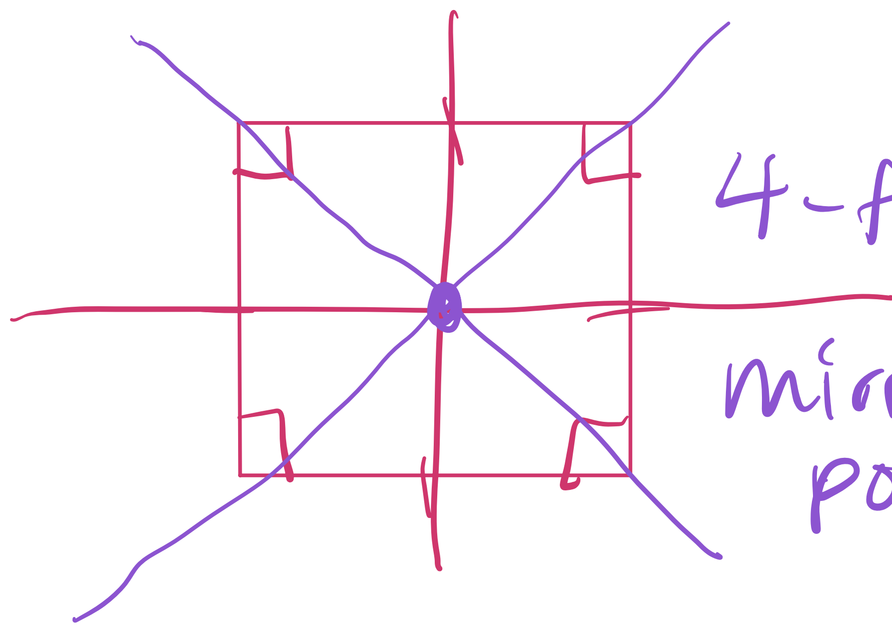
→ mirror lines are fixed by some reflection in the group

* **m-fold mirror points** are points which lie on exactly m mirrors

notice an m-fold mirror point has angle $\frac{m}{\pi}$

The orbifold with mirrorpoints with types a, b, c, ... is denoted *abc...

# Back to tables:

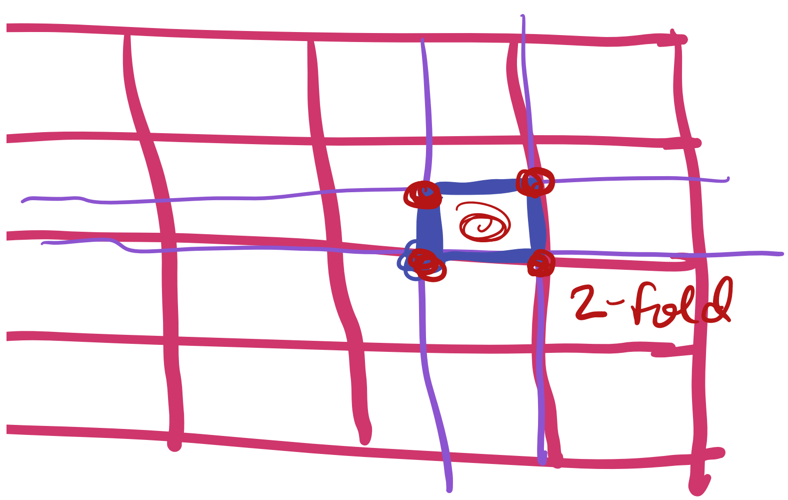


4-fold
mirror
point

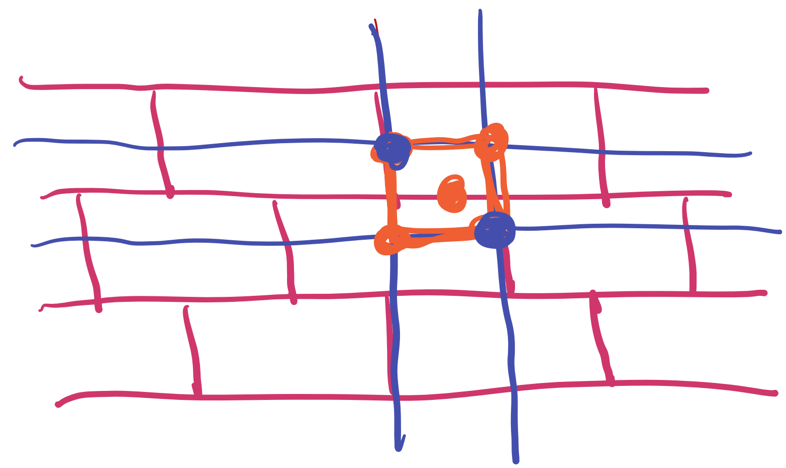
# Understanding Gyrations

A gyration is a point of rotational symmetry which does not lie on any mirror line

*2222




2*22



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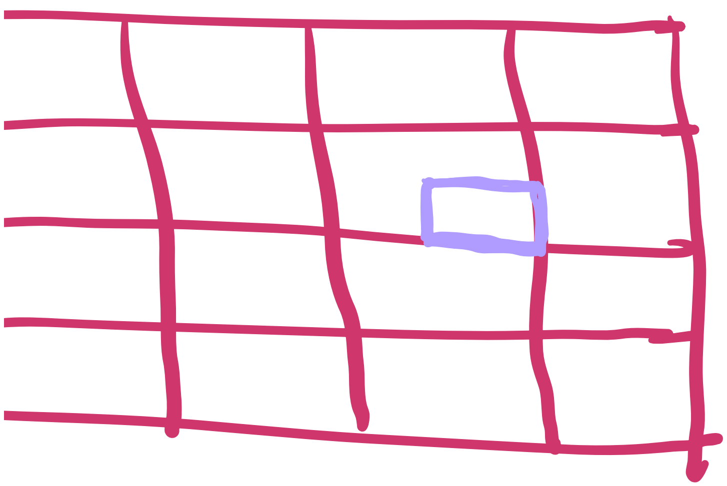

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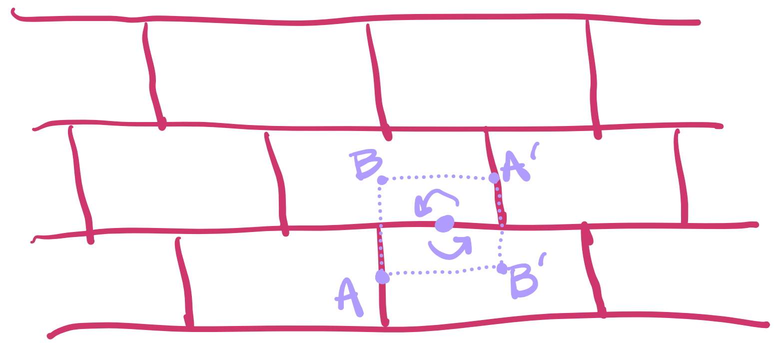

# Orbifold Notation in General

ingredinents:

natural numbers: 1, 2, 3, ...

star * - mirror lines

handle o - wonder

cross cap, × - miracles
guide reflections

of the form

o ... o ABC ... * abc ... * αβγ ... × ... ×

# Euler Characteristic

→ a topological invariant
denoted by $\chi$
classically defined for polyhedra

$$\chi = V - E + F$$

# of vertices (V)
# of edges (E)
# of faces (F)

# Euler Characteristic of Orbifolds

"SymmetryLand Ticket Charges"

$2 to start
(bc sphere)

| ticket type | Symbol | Cost of Ticket Adult | Cost of Ticket Child |
|---|---|---|---|
| 2-trip | 2 | $\frac{1}{2}$ | $\frac{1}{4}$ |
| 3-trip | 3 | $\frac{2}{3}$ | $\frac{1}{3}$ |
| n-trip | n | $\frac{n-1}{n}$ | $\frac{n-1}{2n}$ |
| Top ticket | o or X | 2 | 1 |
| Chaperone's | * | — | 1 |

# SymmetryLand Rules

- → Children without a TOP ticket must have a Chaperone
- → A chaperone's ticket can enter alone or with any number of children
- → Symmetry Land extends credit to regular visitors.

# 17 ways to spend exactly $2

*632
632
*442
442
*333

3*3
333
*2222
2*22
22*
22X

2222
**
*X
XX
O

**Verification** apply the following to a group of characteristic O.

$\boxed{1}$ Replace a group AB…C by *AB…C – this halves the characteristic

$\boxed{2}$ Replace an adult's TOP ticket (o) by two child's ones (x)

$\boxed{3}$ Replace a child's TOP ticket (x) by a chaperone's ticket (*)

$\boxed{4}$ Since a chaperone is now present, replace an adult's n-trip ticket by two child's ones.